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A five-dimensional extension of general relativity and quantum theory

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Abstract
The torsion-free de Sitter reducible metric linear connection on a five-dimensional manifold is studied in conjunction with a method of constructing spacetime via a partially restricted reference cross section of the bundle of linear frames described in earlier publications. It leads to the equation $R_{\alpha\beta} = (1/l)g_{\alpha\beta}$ where l is a fundamental length, assumed to be the Planck length. It also yields the Einstein vacuum equations on the constructed spacetime. Using the Feynman formulation of quantum mechanics together with a dimension reduction by unrestricted transformation of the reference cross section, a geometrical model for quantum non-locality is described. The dimension reduction is then applied to the Schwarzschild solution of Einstein's equations which is found to contain the quantum plane rotation associated with a particle at rest.

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1. Introduction

In this paper, we shall extend the general relativity to five dimensions, repeating the steps that lead to the four-dimensional spacetime. At the same time, we shall use the construction of the observable spacetime according to the procedure described in [1]. The procedure, in a form that suits the subject of the present paper, will be reviewed in section 2. It requires the base manifold to have more than four dimensions, five being the minimum number. At the same time, the de Sitter group that in the five-dimensional setting plays the same role as the Lorentz group in four dimensions must have its Lorentz subgroup clearly identifiable, since the construction relies on the transformations of the reference cross sections of the bundle of linear frames being limited to the Lorentz subgroup. Hence the fifth dimension must be physically distinct from both space and time dimensions in a way similar to what makes time different from space. In other words, it should be associated with a different physical dimension. The obvious choice is mass (energy). It may seem strange to associate mass with

a geometrical dimension, but if the spacetime is a submanifold on which the value of the corresponding coordinate is fixed then there is no reason to exclude such possibility.

The three fundamental physical dimensions, namely length, time and mass, correspond to three fundamental physical constants: the Planck constant $\hbar$, the speed of light c, and the Newton gravitational constant G. The correspondence is via the existence of natural units: the Planck length $l_p = \sqrt{G\hbar/c^3}$, the Planck time interval $\tau_p = \sqrt{G\hbar/c^5}$, and the Planck mass $m_p = \sqrt{c\hbar/G}$.

The step from the Newtonian space and time to the Minkowski spacetime involves multiplication of the points on the time axis by $c = l_p/\tau_p$ to make the physical dimensions of all four components equal, thus enabling the change from the Galileo transformations to the Lorentz pseudo-rotations. Then the Minkowski spacetime is replaced by a four-dimensional spacetime manifold with a torsion-free Lorentz reducible metric linear connection. Finally, a generally covariant set of equations satisfied by the connection is postulated.

We try to repeat the same steps for the third remaining physical dimension, the mass. It means to define the fifth geometrical dimension as the mass axis, multiply its points by $G/c^2 = l_p/m_p$ to achieve the same physical dimension of length for all five geometrical dimensions, and then assume the existence of a five-dimensional manifold with a torsion-free de Sitter reducible metric linear connection. Finally, we should find a suitable generally covariant set of equations satisfied by the connection.

In the next section, we shall review the basic points of [1], applied to the specific case of the bundle of linear frames of a five-dimensional manifold M.

Section 3 will demonstrate how the properties imposed on the geometry of M to guarantee the existence of spacetime lead to a unique set of generally covariant equations, as well as to the vacuum Einstein equations on the constructed spacetime.

Section 4 is devoted to a discussion of quantum non-locality in terms of the Feynman formulation of quantum theory while using further dimension reduction via the unrestricted transformations of the reference cross section.

In section 5, the Schwarzschild solution of Einstein's equations is studied in the five-dimensional setting. The dimension reduction leads to the required quantum rotation $\exp(\mathrm{i}tmc^2/\hbar)$ associated with a particle of mass m at rest.

2. The construction of observable spacetime

The main idea of [1] is that the classical spacetime is a result of observations as opposed to being the fundamental continuum of Nature on which all physical effects take place. Normally, the spacetime observations involve selecting linear frames to form a local cross section of the bundle of linear frames and then measuring the departure of such a cross section from the horizontal direction defined by the linear connection. It is assumed that an observer has an unlimited use of the whole Lorentz group to form the frames. At the same time it is normally understood that the points of the spacetime are directly accessible, that the translations correspond simply to changes of the local coordinates. However, if the structure group is a de Sitter group instead of the Lorentz group, and the local cross section can only be transformed using the Lorentz subgroup, then the translations may be connected with the difference between the cross section and the horizontal direction and measured by the extra rotations and pseudo-rotations. An observer limited to such partially fixed reference cross sections would have a perfect illusion of the classical four-dimensional spacetime without realizing that the linear frame components are at the same time components of a connection. Let us now describe the idea in a concrete mathematical form. While in [1] the discussion was fairly general, here we shall assume from the beginning that we deal with a five-dimensional

manifold equipped with a torsion-free metric linear connection. We use the terminology of the book by Kobayashi and Nomizu [2], but the treatment is more explicit. Local coordinate systems are used more extensively, combined with the use of indices.

We shall use the usual Einstein's summation convention as well as the following conventions regarding the role and the range of the indices. Greek indices are used for the local coordinates, while the Latin indices are reserved for the orthonormal linear frames. Both kinds of the indices taken from the beginning of the alphabet ($\alpha, \beta, \gamma, \delta$ and a, b, c, d) range from 1 to 5, while indices μ, ν, ρ, σ and i, j, k, l range from 1 to 4.

Let M be a five-dimensional manifold covered by neighbourhoods with local coordinates x^α. The physical dimension of all five coordinates is the length. We assume that there is a metric tensor $g_{\alpha\beta}(x)$ defined on M. Although it is very likely that a discussion of the global structure will play a key role in further development, in the present paper we deal only with the local description.

There is a choice of signatures for the de Sitter group, namely (4, 1) or (3, 2). At this stage, either of them could be used, but later we shall see that a link to quantum theory described in section 4 requires the (3, 2) signature. We define the orthogonal frames

$$h_a^\alpha(x)\frac{\partial}{\partial x^\alpha}$$

by

$$h_a^\alpha(x)h_b^\beta(x)g_{\alpha\beta}(x) = g_{ab} = \mathrm{diag}(1, 1, 1, -1, -1).$$

The manifold which is obtained by extending M to include all possible linear frames at every point is called the bundle of linear frames of M. It is a principal fibre bundle with M as its base manifold. Once a local cross section $h_a^\alpha(x)$ is chosen, a point in the bundle of linear frames is represented by a pair (x, g), where $x \in M$ and $g \in G$, the de Sitter group. The connection can then be explicitly given by the horizontal lift of $\partial/\partial x^\alpha$:

$$X_\alpha^{(h)} = \frac{\partial}{\partial x^\alpha} - A_{\alpha b}^a(x)L_a^b(g)$$

where $L_a^b(g)$ represent elements of the 5×5 general linear Lie algebra and $A_{\alpha b}^a(x)$ are the connection components. Of course, reducible connections are defined by only ten independent components

$$A_\alpha^{ab} = A_{\alpha c}^a g^{cb} = -A_\alpha^{ba}.$$

The curvature is defined via the commutator

$$\left[X_\beta^{(h)}, X_\alpha^{(h)}\right] = \left(\partial_\alpha A_{\beta b}^a - \partial_\beta A_{\alpha b}^a + A_{\alpha c}^a A_{\beta b}^c - A_{\beta c}^a A_{\alpha b}^c\right)L_a^b(g) = R_{b\alpha\beta}^a L_a^b(g). \tag{1}$$

When the reference cross section is changed via multiplication by an x-dependent element of the de Sitter group defined by its matrix elements $p_b^a(x)$ the frame components transform as

$$\tilde{h}_\alpha^a = p_b^a h_\alpha^b \tag{2}$$

and the connection components transform according to

$$\tilde{A}_{\alpha b}^a = p_c^a A_{\alpha d}^c q_b^d - \left(\partial_\alpha p_c^a\right)q_b^c \tag{3}$$

where

$$p_c^a g_{ab} p_d^b = g_{cd}$$

and q_b^a are the elements of the matrix inverse to p_b^a [3]. When the reference cross section is selected simply as $\{\partial/\partial x^\alpha, \alpha = 1, \ldots, 5\}$, the connection components are called Christoffel

symbols and denoted by $\Gamma^{\gamma}_{\alpha\beta}$. The transformation between the Γ's and the A's is similar to equation (3) with h^a_{α} and its matrix inverse h^{α}_a replacing p's and q's:

$$\Gamma^{\gamma}_{\alpha\beta} = h^{\gamma}_a A^a_{\alpha b} h^b_{\beta} - \left(\partial_{\alpha} h^{\gamma}_a\right) h^a_{\beta} \tag{4}$$

or

$$A^a_{\alpha b} = h^a_{\gamma} \Gamma^{\gamma}_{\alpha\beta} h^{\beta}_b - \left(\partial_{\alpha} h^a_{\gamma}\right) h^{\gamma}_b. \tag{5}$$

We shall now list the conditions that lead to the construction of an observable four-dimensional spacetime when the transformations of the reference cross section are restricted to the Lorentz subgroup. As was mentioned before, the restriction is assumed to be due to limited methods used in spacetime observations. It is also important to realize that the Lorentz subgroup must be clearly identified. The restriction would not be well defined if the extra axis was just another spatial or temporal dimension. This makes mass the likely candidate for the physical dimension associated with the extra axis.

We define the fifth coordinate by

$$A^a_{5b} = 0 \tag{6}$$

and a four-dimensional submanifold N of M by $x^5 = $ const. When the fifth dimension is given the meaning of mass, equation (6) could be considered as a geometrical expression for the law of conservation of energy.

An observer is assumed to investigate the geometry of M and measure $A^i_{\mu j}$ as well as $A^i_{\mu 5}$. When the transformations of the reference cross section are limited to the Lorentz subgroup associated with spacetime, the transformations of $A^i_{\mu 5}$ and h^i_{μ} are identical. Indeed, the restriction means that

$$p^i_5 = p^5_i = 0, \qquad p^5_5 = 1$$

and equation (3) then splits into

$$\tilde{A}^i_{\mu j} = p^i_k A^k_{\mu l} q^l_j - \left(\partial_{\mu} p^i_k\right) q^k_j \tag{7}$$

and

$$\tilde{A}^i_{\mu 5} = p^i_j A^j_{\mu 5}. \tag{8}$$

Components $A^i_{\mu 5}$ characterize the difference between the reference cross section and the horizontal direction that is outside the Lorentz subalgebra. For the observer they would appear as the frame components associated with translations. Since the physical dimension of $A^i_{\mu 5}$ is $(\text{length})^{-1}$ while h^i_{μ} are dimensionless, we write

$$l A^i_{\mu 5} = h^i_{\mu} \tag{9}$$

where l is a unit of length.

From equation (1) we have

$$R^i_{5\mu\nu} = \partial_{\mu} A^i_{\nu 5} - \partial_{\nu} A^i_{\mu 5} + A^i_{\mu j} A^j_{\nu 5} - A^i_{\nu j} A^j_{\mu 5}.$$

Using equations (4) and (9) we get

$$R^i_{5\mu\nu} = \frac{1}{l} h^i_{\sigma} \left(\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\nu\mu}\right).$$

In the usual approach that uses the canonical form and the affine bundle, this is the formula connecting the torsion with the curvature components corresponding to translations (see [[2], p 130]). The observer that is unable to use the full de Sitter transformations would not be aware that the true nature of the group elements outside the Lorentz subgroup is rotations and pseudo-rotations and not translations.

There is, however, one important difference between the usual affine bundle approach and the de Sitter case. The curvature of the observed spacetime is

$$R^{i(o)}_{j\mu\nu} = \partial_\mu A^i_{\nu j} - \partial_\nu A^i_{\mu j} + A^i_{\mu k} A^j_{\nu j} - A^i_{\nu k} A^k_{\mu j}$$

but the full curvature from equation (1) yields

$$R^i_{j\mu\nu} = R^{i(o)}_{j\mu\nu} + \frac{1}{l^2} \left(h^i_\mu h^k_\nu - h^i_\nu h^k_\mu \right) g_{kj}.$$

Even for the flat Minkowski space characterized by $R^{i(o)}_{j\mu\nu} = 0$ the curvature is not zero. Under the Lorentz transformations $R^i_{j\mu\nu}$ and $R^i_{5\mu\nu}$ transform separately, but outside the Lorentz subgroup they mix. When the torsion of the observed spacetime is zero, it stays zero only when no transformations outside the Lorentz subgroup are performed. This provides a strong hint concerning the reasons for the restriction of the transformations. Spacetime geometry is reconstructed basically from observing paths of photons and other free particles, i.e. from geodesics. The procedure for constructing connections from geodesics (Schild's ladder) is known to produce automatically connections with zero torsion (see [[4], p 248]).

3. The fundamental equations

In this section, we shall find the generally covariant equations that the torsion-free metric linear connection on M must satisfy in order to make the construction of the observable spacetime possible.

The condition we have to impose is that for some reference cross section equations (6) and (9) are satisfied. For simplicity we shall work with a reference cross section that separates x^5 from the remaining four coordinates:

$$h^5_5 = 1, \qquad h^i_5 = 0, \qquad h^5_\mu = 0 \tag{10}$$

for all values of i and μ. Of course, the resulting equations will not depend on any particular choice of the reference cross section.

From the components $A^a_{\alpha b}$ satisfying equations (6) and (9), we can compute the Christoffel symbols using equation (4):

$$\Gamma^\sigma_{\mu\nu} = h^\sigma_i A^i_{\mu j} h^j_\nu - \left(\partial_\mu h^\sigma_i \right) h^i_\nu$$

$$\Gamma^\sigma_{\mu 5} = \frac{1}{l} h^\sigma_i h^i_\mu = \frac{1}{l} \delta^\sigma_\mu$$

$$\Gamma^\sigma_{5\mu} = - \left(\partial_5 h^\sigma_i \right) h^i_\mu.$$

Thus the condition of zero torsion yields

$$\partial_5 h^\sigma_i = -\frac{1}{l} h^\sigma_i,$$

which leads to

$$h^i_\mu(x^1, x^2, x^3, x^4, x^5) = e^{x^5/l} h^i_\mu(x^1, x^2, x^3, x^4, 0), \tag{11}$$

$$g_{\mu\nu}(x^1, x^2, x^3, x^4, x^5) = e^{2x^5/l} g_{\mu\nu}(x^1, x^2, x^3, x^4, 0), \tag{12}$$

as well as

$$g_{\mu 5} = 0, \qquad g_{55} = -1. \tag{13}$$

Further,

$$\Gamma^5_{\mu\nu} = h^5_5 A^5_{\mu i} h^i_\nu = g_{ij} A^j_{\mu 5} h^i_\nu = \frac{1}{l} g_{ij} h^j_\mu h^i_\nu = \frac{1}{l} g_{\mu\nu}.$$

To summarize, while

$$\Gamma^{\sigma}_{\mu\nu} = \tfrac{1}{2} g^{\sigma\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}), \tag{14}$$

the Γ's containing a single index 5 are

$$\Gamma^5_{\mu\nu} = \frac{1}{l} g_{\mu\nu}, \qquad \Gamma^\nu_{\mu 5} = \Gamma^\nu_{5\mu} = \frac{1}{l} \delta^\nu_\mu. \tag{15}$$

Γ's containing more than one index 5 are all equal to zero.

Using equations (11)–(15) to calculate the components of the Ricci tensor according to the formula

$$R_{\alpha\beta} = \partial_\gamma \Gamma^\gamma_{\alpha\beta} - \partial_\beta \Gamma^\gamma_{\alpha\gamma} + \Gamma^\delta_{\delta\gamma} \Gamma^\gamma_{\alpha\beta} - \Gamma^\delta_{\alpha\gamma} \Gamma^\gamma_{\beta\delta}$$

we obtain

$$R_{\mu 5} = R_{5\mu} = 0 \qquad R_{55} = -\frac{4}{l^2} = \frac{4}{l^2} g_{55}$$

and

$$R_{\mu\nu} = R^{(o)}_{\mu\nu} + \frac{4}{l^2} g_{\mu\nu},$$

where $R^{(o)}_{\mu\nu}$ denotes the Ricci tensor of the observed four-dimensional spacetime N.

Such equations can be replaced by a single generally covariant tensor equation

$$R_{\alpha\beta} = \frac{4}{l^2} g_{\alpha\beta} \tag{16}$$

on M, provided that

$$R^{(o)}_{\mu\nu} = 0. \tag{17}$$

Equation (16) is the generally covariant equation we were seeking, while equation (17) must be satisfied by the special class of solutions characterized by the fact that they can lead to the construction of a classical spacetime. We regard equation (16) as the fundamental equation describing the geometry of the universe, while the solutions satisfying equations (12), (13) and (17) describe the geometry of the macroscopic spacetime.

The unit l plays no role in the geometry of the constructed spacetime, and its value is arbitrary. However, it is attractive to think of it as the natural unit, namely the Planck length. The full five-dimensional theory then contains the Planck constant and may, in principle, show certain links to quantum theory. In addition, l plays the role of the radius for the solutions of equation (16) such as the constant curvature solution. Such solutions may be connected with the microscopic geometrical structures and the existence of elementary particles.

4. Non-locality in quantum theory

It is well known that quantum theory exhibits features that are difficult to accommodate within the classical spacetime. A single particle may take different paths and interfere with itself, or pairs of coherent particles may keep the coherence even when they are separated by macroscopic distances, etc. An extensive description of the effects as well as various attempts at their explanation can be found for example in a book by Grib and Rodriguez [5]. It is important to realize that particles exhibit non-locality only when they remain unobserved. In our approach based on a construction of spacetime via a partially fixed reference cross section we assumed that the cross section is selected at every point within a region of the five-dimensional manifold. However, if there are regions where no observation is carried out, the reference cross section in such regions may be a result of certain interpolation and

it may not have the properties required for the existence of a spacetime structure. With the quantum non-locality in mind one is interested in reference cross sections which may follow the horizontal direction also in other dimensions in addition to x^5, thus eliminating spatial separation. It was shown in [1] that when the flat Minkowski spacetime is generated from the (3, 2) de Sitter connections, spatial dimensions can indeed be eliminated, leaving only the time as an observable dimension. The remaining connection component is then $A_{t5}^4 = 1/l$.

Connection component A_{t5}^4 has the same value of $1/l$ also before any dimension reduction when Minkowski coordinates and the corresponding Lorentz frames are used. The discrepancy between the horizontal lift of a curve and its lift onto the reference cross section is interpreted as time translation by an observer unable to perform transformations outside the Lorentz subgroup. Under any Lorentz transformations the value of A_{t5}^4 changes as expected, i.e. according to the transformation of the frame components h_μ^i. The rotation responsible for the discrepancy remains hidden. Similarly, when the spacetime is deformed due to the presence of matter, the value of A_{t5}^4 before the dimension reduction is just as expected within the classical relativity. After the dimension is reduced, the difference between the value of A_{t5}^4 and $1/l$ characterizes the presence of matter. As $A_{t5}^4 = -A_{t4}^5$, the difference leads to an additional rotation within the (4, 5)-plane. When we look for its physical manifestation, we have to look beyond spacetime geometry. The group of plane rotations can be identified with the multiplicative group of complex numbers with modulus 1 via identification of the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

with the imaginary unit i. This leads to the possibility of the quantum rotation $\exp(itmc^2/\hbar)$ associated with a particle at rest being a manifestation of the additional rotation caused by the presence of matter and to the interpretation of the (4, 5)-plane as a geometrical model for the complex plane of quantum theory.

The transformation of the reference cross section that eliminates the spatial dimensions can be considered as a map from N onto the bundle manifold Q of a principal fibre bundle $Q(T, U(1))$, where T is a one-dimensional base manifold and $U(1)$ is the group of de Sitter rotations within the (4, 5)-plane [6]. Any time-like curve on N is mapped into a cross section of Q, using the proper time measured along the curve as a coordinate on T. The time axis itself maps into the cross section characterized by $\exp(it/l)$, while other time-like curves yield

$$\exp\left(\frac{i}{l} \int_0^t \frac{d\tau}{\sqrt{1 - u^2(\tau)}}\right)$$

where $u(\tau) = v(\tau)/c$. The transformation connecting the two cross sections is then characterized by $g(t) \in U(1)$ written as

$$g(t) = \exp\left[\frac{i}{l} \int_0^t \left(\frac{1}{\sqrt{1 - u^2(\tau)}} - 1\right) d\tau\right]. \tag{18}$$

The main reason for the investigation of the dimension reduction is its possible relationship to the non-locality of quantum mechanics. Ultimately, one should incorporate the notion of the reference cross section and its transformations into the theory. For the sake of further research it is useful to look at various existing formulations of quantum mechanics to see if there is one with a mathematical structure displaying some features that are common with the above-described procedure. From what follows it can be seen that the prime candidate is the Feynman's spacetime formulation based on path integrals [7].

Consider a free particle travelling from point A at time t_A to a point B at time t_B. According to Feynman's formalism, the corresponding probability amplitude is given by

$$\sum \exp\left(\frac{i}{\hbar}\int_{t_A}^{t_B} E_{kin} dt\right) \qquad (19)$$

where each integral is taken along a path form A to B and the sum is over all such possible paths. Writing E_{kin} as

$$E_{kin}(\tau) = mc^2\left(\frac{1}{\sqrt{1-u^2(\tau)}} - 1\right)$$

where m is the rest mass of the particle brings the expression (19) into the form

$$\sum \exp\left[i\frac{mc^2}{\hbar}\int_{t_A}^{t_B}\left(\frac{1}{\sqrt{1-u^2(\tau)}} - 1\right) d\tau\right]. \qquad (20)$$

The particle paths correspond to cross sections of Q and are related to that describing the particle at rest by the same kind of transformation as in equation (18). However, the particle at rest corresponds to $\exp(it/l_m)$ where $l_m = \hbar/mc^2$ instead of $\exp(it/l)$ for the Minkowski time axis. Does this fact also have a geometrical origin? If it does, then it should be found in the geometry of the spacetime surrounding the particle of mass m at rest. In particular, one should attempt to eliminate spatial coordinates in the Schwarzschild solution of Einstein's equations and then consider the remaining connection component A_{t5}^4 at a minimum value of the radial coordinate. The transformation from the cross section characterized by $A_{t5}^4(r_{min})$ to the Minkowski axis reads as

$$\exp\left[it\left(\frac{1}{l} - A_{t5}^4(r_{min})\right)\right].$$

The question is whether the Schwarzschild geometry leads to

$$\frac{1}{l} - A_{t5}^4(r_{min}) = \frac{mc^2}{\hbar}. \qquad (21)$$

5. Dimension reduction for the Schwarzschild geometry

In this section, we shall consider the solution of equation (16) that leads to the Schwarzschild geometry of the constructed observable spacetime. Using equation (3) in its matrix form

$$\tilde{A}_\alpha = P A_\alpha P^{-1} - (\partial_\alpha P)P^{-1}. \qquad (22)$$

we shall eliminate completely the fifth column and the fifth row of A_r and then perform additional transformation to reduce the nonzero elements of A_t to A_{t5}^4 and A_{t4}^5 only. This yields a unique expression for the difference $1/l - A_{t5}^4(r)$.

The four-dimensional spacetime describing the external region of a spherically symmetric particle of mass m in the polar spherical coordinates has the metric given by (see p 607 of [4])

$$g_{rr} = \left(1 - \frac{r_o}{r}\right)^{-1}, \qquad g_{\theta\theta} = r^2, \qquad g_{\phi\phi} = r^2\sin^2\theta, \qquad g_{tt} = -\left(1 - \frac{r_o}{r}\right) \qquad (23)$$

where r_o is the Schwarzschild radius

$$r_o = 2Gm$$

with $c = 1$. The nonzero Christoffel symbols are then calculated as

$$\Gamma^r_{rr} = -\frac{r_o}{2r(r-r_o)}, \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \qquad \Gamma^t_{rt} = \Gamma^t_{tr} = \frac{r_o}{2r(r-r_o)}$$

$$\Gamma^r_{\theta\theta} = -(r-r_o), \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta}, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^r_{tt} = \frac{r_o(r-r_o)}{2r^3}, \qquad \Gamma^t_{tr} = \Gamma^t_{rt} = \frac{r_o}{2r(r-r_o)}.$$

To obtain the spacetime connection components we select a cross section in the bundle of linear frames in the form

$$H = [h^i_\mu] = \begin{bmatrix} \frac{\sin\theta\cos\phi}{\sqrt{1-\frac{r_o}{r}}} & r\cos\theta\cos\phi & r\sin\theta\sin\phi & 0 \\ \frac{\sin\theta\sin\phi}{\sqrt{1-\frac{r_o}{r}}} & r\cos\theta\sin\phi & -r\sin\theta\cos\phi & 0 \\ \frac{\cos\theta}{\sqrt{1-\frac{r_o}{r}}} & -r\sin\theta & 0 & 0 \\ 0 & 0 & 0 & \sqrt{1-\frac{r_o}{r}} \end{bmatrix} \tag{24}$$

and use equation (5) written as

$$A_\mu = H\Gamma_\mu H^{-1} - (\partial_\mu H)H^{-1}. \tag{25}$$

This results in the connection components A_r, A_θ, A_ϕ and A_t as 4×4 matrices. A_r has all matrix elements equal to zero, while

$$A_t = \begin{bmatrix} 0 & 0 & 0 & \frac{r_o}{2r^2}\sin\theta\cos\phi \\ 0 & 0 & 0 & \frac{r_o}{2r^2}\sin\theta\sin\phi \\ 0 & 0 & 0 & \frac{r_o}{2r^2}\cos\theta \\ \frac{r_o}{2r^2}\sin\theta\cos\phi & \frac{r_o}{2r^2}\sin\theta\sin\phi & \frac{r_o}{2r^2}\cos\theta & 0 \end{bmatrix}.$$

Using equation (9) we can extend A_r and A_t to 5×5 matrices. At $x^5 = 0$

$$A_r = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{\sin\theta\cos\phi}{l\sqrt{1-\frac{r_o}{r}}} \\ 0 & 0 & 0 & 0 & \frac{\sin\theta\sin\phi}{l\sqrt{1-\frac{r_o}{r}}} \\ 0 & 0 & 0 & 0 & \frac{\cos\theta}{l\sqrt{1-\frac{r_o}{r}}} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\sin\theta\cos\phi}{l\sqrt{1-\frac{r_o}{r}}} & \frac{\sin\theta\sin\phi}{l\sqrt{1-\frac{r_o}{r}}} & \frac{\cos\theta}{l\sqrt{1-\frac{r_o}{r}}} & 0 & 0 \end{bmatrix}$$

and

$$A_t = \begin{bmatrix} 0 & 0 & 0 & \frac{r_o}{2r^2}\sin\theta\cos\phi & 0 \\ 0 & 0 & 0 & \frac{r_o}{2r^2}\sin\theta\sin\phi & 0 \\ 0 & 0 & 0 & \frac{r_o}{2r^2}\cos\theta & 0 \\ \frac{r_o}{2r^2}\sin\theta\cos\phi & \frac{r_o}{2r^2}\sin\theta\sin\phi & \frac{r_o}{2r^2}\cos\theta & 0 & \frac{1}{l}\sqrt{1-\frac{r_o}{r}} \\ 0 & 0 & 0 & -\frac{1}{l}\sqrt{1-\frac{r_o}{r}} & 0 \end{bmatrix}.$$

The dimension reduction via an unrestricted transformation of the reference cross section may now be performed.

With

$$
P = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
$$

we obtain

$$
\tilde{A}_r = \begin{bmatrix}
0 & 0 & 0 & \frac{\sin\theta\cos\phi}{l\sqrt{1-\frac{r_o}{r}}} & 0 \\
0 & 0 & 0 & \frac{\sin\theta\sin\phi}{l\sqrt{1-\frac{r_o}{r}}} & 0 \\
0 & 0 & 0 & \frac{\cos\theta}{l\sqrt{1-r_o r}} & 0 \\
\frac{\sin\theta\cos\phi}{l\sqrt{1-\frac{r_o}{r}}} & \frac{\sin\theta\sin\phi}{l\sqrt{1-\frac{r_o}{r}}} & \frac{\cos\theta}{l\sqrt{1-\frac{r_o}{r}}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

and

$$
\tilde{A}_t = \begin{bmatrix}
0 & 0 & 0 & 0 & -\frac{r_o}{2r^2}\sin\theta\cos\phi \\
0 & 0 & 0 & 0 & -\frac{r_o}{2r^2}\sin\theta\sin\phi \\
0 & 0 & 0 & 0 & -\frac{r_o}{2r^2}\cos\theta \\
0 & 0 & 0 & 0 & \frac{1}{l}\sqrt{1-\frac{r_o}{r}} \\
-\frac{r_o}{2r^2}\sin\theta\cos\phi & -\frac{r_o}{2r^2}\sin\theta\sin\phi & -\frac{r_o}{2r^2}\cos\theta & -\frac{1}{l}\sqrt{1-\frac{r_o}{r}} & 0
\end{bmatrix} .
$$

To achieve A_{t5}^4 and A_{t4}^5 as the only remaining nonzero matrix elements of A_t it is enough to use a spatial rotation that transforms the last column into

$$
\begin{bmatrix}
0 \\
0 \\
\frac{r_o}{2r^2} \\
\frac{1}{l}\sqrt{1-\frac{r_o}{r}} \\
0
\end{bmatrix}
$$

and finally a Lorentz transformation that yields

$$
\tilde{A}_{t5}^4 = \sqrt{\frac{1}{l^2}\left(1-\frac{r_o}{r}\right) - \frac{r_o^2}{4r^4}}. \tag{26}
$$

The expression for A_{t5}^4 in equation (26) determines how time is observed via the rotation within the (4, 5)-plane of the orthonormal frame when it is the only remaining observable dimension. But is this expression unique? Let us investigate how it can be changed by a transformation of the reference cross section, while it remains the only nonzero element within the fifth column of A_t. Spatial rotations would not change its value, while any other Lorentz transformation would introduce nonzero elements into the fifth column of A_t. However, a transformation involving only rotations within the (4, 5)-plane could change its value when equation (22) is used. Not a constant transformation, but a variable transformation would bring the additive term $(\partial_t P)P^{-1}$. This does not constitute any difficulty, since we are after the value of the left-hand side of equation (21) and the term would be added to both the vacuum value $1/l$ as well as to A_{t5}^4.

As expected, $A_{t5}^4 \to 1/l$ as $r \to \infty$, since far from the origin the geometry approaches that of a flat spacetime. Of course, $A_{t5}^4(r) = 0$ for any value of r when $m = 0$.

The minimum value of r in equation (21) should correspond to something like a classical radius of the particle. It should mark the boundary between the internal region and the external classical spacetime. At this stage there is no precise prescription how to choose $r_{\min}$, except that in equation (16) l is a radius associated with the particle-like solutions. However, the most compelling argument for $r_{\min} = l = l_p$ is the fact that

$$\frac{1}{l} - \sqrt{\frac{1}{l^2}\left(1 - \frac{r_o}{l}\right) - \frac{r_o^2}{4l^4}} \doteq \frac{r_o}{2l^2} = \frac{m}{\hbar}.$$

The approximation is based on $r_o \ll l$ which is quite well satisfied for elementary particles. For example, the electron has $r_o/l \sim 10^{-22}$.

6. Summary and perspectives

In the present paper, consequences of three assumptions were discussed.

(a) That the fundamental manifold is five-dimensional with geometrical properties that parallel those of the four-dimensional spacetime in the classical general relativity.
(b) That the change of position in the five-dimensional manifold is measured by the difference between the reference cross section in its bundle of linear frames and the horizontal direction defined by the linear connection.
(c) That the classical spacetime observations restrict the transformation of the reference cross section to the Lorentz subgroup consisting of spacetime rotations and pseudo-rotations.

The consequences demonstrated in the present paper are as follows.

(1) The Ricci tensor on the five-dimensional manifold is equal to $1/l$ times the metric tensor, where l is a unit of length.
(2) The Ricci tensor on the constructed four-dimensional spacetime is equal to zero.
(3) Unrestricted transformations of the reference cross section can be used to eliminate the spatial dimensions not only for the Minkowski flat spacetime, but also for the Schwarzschild geometry.
(4) Temporal translations are measured by ordinary rotations providing a geometrical model for the quantum complex plane.
(5) After the maximal dimension reduction the Schwarzschild geometry yields at $r = l$ the rotation $\exp(\mathrm{i}tmc^2/\hbar)$ when l is the Planck length and m is the mass of the particle located at the origin.

The consequences of the geometrical scheme described above suggest that it may have something to do with the actual geometrical structure of the universe, but it could also be just an interesting mathematical construction and the results connecting it to the real world a mere coincidence. The truth can be discovered only by further research. Since we are in an unchartered territory it is not easy to guess what direction the research should take. However, several topics stand out as obvious candidates.

The relationship between the dimension reduction and quantum theory needs deeper investigation. If the spacetime observations determine a specific cross section of the bundle of frames, then an interpolation should be used in the regions where no spacetime observations are carried out. The resulting cross section should form a part of a reformulated quantum theory.

The dimension reduction by the transformation of the reference cross section should be applied to more complicated solutions of Einstein's equations. The Kerr solution is the obvious next choice.

Apart from the solutions of equation (16) that lead to the observable spacetime satisfying Einstein's vacuum equations, other solutions should be investigated. In particular, the microscopic particle-like globally non-trivial solutions. While in the present paper the interpretation of the fifth dimension as a geometrical from of the third physical dimension was never really used in any calculation, it may play a significant role in the analysis of such solutions.

Finally, impact on the theory of quantum gravity may be significant. The classical gravity needs the reference cross section generating the spacetime with Einstein's geometry to be defined everywhere. If a correct quantum theory needs a departure from such requirement, then it is not surprising that so many attempts to quantize gravity failed.

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